Rough-Calculus
and
Numerical Analysis-
A Mathematical Foundation

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Good Morning Everyone!

Let us Greet
Professor Skowron
A Very Happy Birthday!

Presentation of Two Theories

1) A Proposed Theory for RC, Rough Calculus (1995)

2) A Mathematical Theory of Numerical Analysis

In his second talk for 1995 Lotfi A. Zadeh Best Paper Award, Professor Pawlak presented a very distinguished idea in rough sets, called

Rough Calculus

What is RC (ROUGH CALCULUS)?

1) A family of rough sets whose total approximation space is the real line (on which College Calculus can live in).

2) Our RC is depended on our solution of the problem stated in a text of

Numerical Analysis

1. by L. Ridgway Scott

(Princeton Univ. Press 2016)

2. at Third Bullet (Ch.1, page 1)

- the effects of finite-precision arithmetic (a.k.a. round-off error).

The first of these just means that the algorithm approximates the desired quantity to any required accuracy under suitable restrictions. The second means that the behavior of the algorithm is continuous with respect to the parameters of the algorithm. The third topic is still not well understood at the most basic level, in the sense that there is not a well-established mathematical model for finite-precision arithmetic. Instead, we are forced to use crude upper bounds for the behavior of finite-precision arithmetic that often lead to overly pessimistic predictions about its effects in actual computations.

The Solution of this Bullet will give us a good theory of Pawlak's RC:

- An extended abstract for numerical analysis has been published in the Encyclopedia Complexity and System Science (March 2023) that needs some updating.
- A pure math version is in the process of submitting to Math Journal.

Traditional Approximation-theory

A meterstick has markings for

(1) centimeters,

(2) millimeters, and one estimated markings. Let us call it

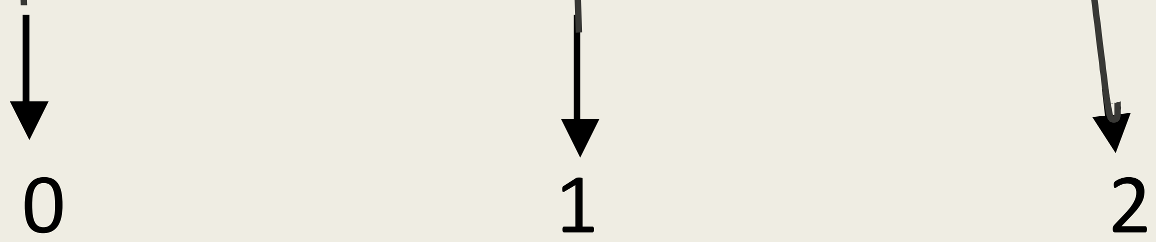
(3) fine-meters.

Centimeter-marking induces

1st Partition of the Real Line:

[-0.5, 0.5), [0.5, 1.5), [1.5, 2.5), . . .

By metersticks, a point in a interval given above is mapped to the mid-point of each interval.



Millimeter-marking induces

2nd Partition on the Real Line:

[-0.05, 0.05), [0.95, 1.05), [1.95, 2.05).

By metersticks, a point in a interval given above is mapped to the mid-point of each interval.

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0.0 1.0 2.0

Fine-meter Marking induces

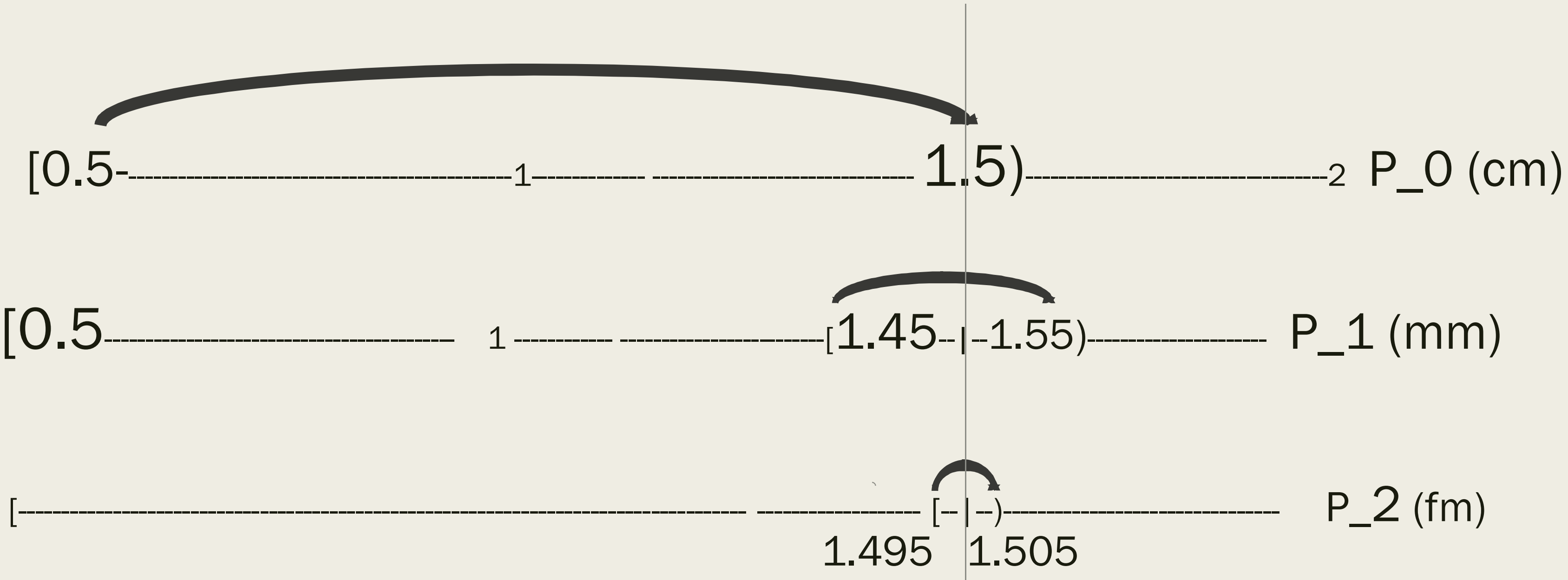
3rd Partition on the Real Line:

[-0.005, 0.005), [0.995, 1.005), [1.995, 2.005),

By metersticks, a point in a interval given above is mapped to the mid-point of each interval.

0.00 1.00 2.00

3 Levels of Markings induce 3 (non-refined) Partitions



A typical left-closed right-open interval (an equivalence class) is displayed in each partition.

Approximation-theory

As illustrated, three markings induces three levels of partitions that induces a new partition.

Proposition 1:

(1) All possible intersections of all these equivalence classes form a new partition, denoted by

$$P_0 \cap P_1 \cap P_2.$$

(2)The topology generated by $P_0 \cap P_1 \cap P_2$ is called Pawlak topology.

Pawlak Topology

A topology is a family T of sets which satisfies the two Conditions: the intersection of any two members of T is a member of T, and the union of the members of each subfamily of T is a member of T.

Let (U, R) be a Rough set, where U is a set and R is an equivalence relation on U. R defines a partition P on U. Then the power set (which includes empty set) of P defines a topology, called Pawlak Topology.

Granular Topology of the Reals

A real line R, as a generalized meterstick, has ∞-ly many markings on n-decimal digits marking, n=0, 1, 2, … So measuring with such a line induces ∞ levels of partitions, denoted by P_∞.

Let the family G_∞ be the family of all finite intersections of members of P_∞.

Such G_∞ is a topology, called granular topology, of the reals.

Granular Topology of a Family P∞ (Kelley)

"If P∞ is any family of the sets the family of all finite intersections of members of P∞ is the base G∞ for a topology for the set $X = \cup\{ S \mid S \in P\infty\}$."

Observe P_∞ and G_∞ are special

P∞ and G∞ (when X=R). So G_∞ is RC.

Granular Topology of a Family P∞ (Spanier)

Let X and Xn (n is an Einstein notation) be topological spaces defined by the P∞ or G∞, respectively. Then, "the topology induced on X by functions {hn : X −→ Xn} is characterized by the property that if Y is a topological space, a function: g : Y −→ X is continuous if and only if hn ◦ g : Y −→ Xn is continuous for each n. "

Granular to Usual Topologies

The identity map induces a continuous map from the real line with granular topology to the real line with usual topology.

The inverse of the identity map

The inverse of the identity map induces a sub-topology of the granular topology that is isomorphic to the usual topology.

Cor: This Theorem implies RC

A Revised Proposition from

(the Book of Foundations)

"We can show here the existence of $m\_*(x\infty) \oplus m\_*(y\infty) \in R$ such that for any p = 0, 1, 2 . . . there exists an n(p) (which, we will assume, is the smallest integer) such that, for n > n(p), $m\_n(x\infty) + m\_n(y\infty)$ and $x\infty + y\infty$ have the same integral part and have the same digits in their first p decimal places"

Granular Ordered Field

$$m_*^U : (\mathcal{R}, +, \times, >) \cong (\mathcal{D}, \oplus, \otimes, >_1)$$

The right-tuple represents the complete ordered field $\mathcal{D}$ of Measurement Vectors, and the left-tuple represents the usual complete ordered field the reals, $\mathcal{R}$